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Letter to the Editor

An improved criterion of Gaussian equivalent linearization for analysis of non-linear stochastic systems

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1. Introduction

The criteria for constructing an equivalent linear system are usually based on the minimization of some specific deviation measure. This technique was first developed for deterministic non-linear problems. Caughey [1] adapted this technique to apply to stochastic systems. The standard way of implementing this technique is to minimize a mean square measure of the difference between the non-linear and the equivalent linear equations. Gaussian equivalent linearization (GEL) proposed by Caughey is presently the simplest tool widely used for analysis of non-linear stochastic problems because GEL allows to use up the available analytic results from stochastic linear systems. There are many other approximate methods such as moment closure, equivalent non-linear equation, Markov methods, Monte Carlo simulation, etc., which are best suited for simple-degree-of-freedom systems, with stationary random excitation. For multi-degree-of-freedom (MDOF) systems, which are prevalent in most engineering applications, these methods are very difficult to apply; they tend to involve severe analytical complexity, often combined with excessive computational requirements, in terms of core storage or execution time, and in general extremely costly. The single exception is GEL, which enables results to be obtained with relative ease, even in situations where MDOF systems subjected to non-stationary random excitations are of concern. However, a major limitation of GEL is perhaps that its accuracy decreases as the non-linearity increases, and for many cases it can leads to unacceptable errors. Therefore, a series of researches of improving GEL has been done for the past some decades by many authors (see, e.g., Refs. [2–13]).

An alternative extension of GEL has recently been proposed by Anh and Di Paola [14]. This extension is refereed to as 'local mean square error criterion' (LOMSEC). The Authors gave initial tests based on Duffing and Vanderpol oscillators under a zero mean Gaussian white noise. Following the initial efforts of Anh and Di Paola, Hung examined the proposed technique

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through analysis of a series of diversely various non-linear random systems such as the analysis of the response moments of simple-degree-of-freedom (SDOF) systems [15,16], the analysis of the mean up-crossing rate [17] and the exceedance probability of response [18].

The results obtained from the above mentioned researches show advance of LOMSEC, especially the accuracy of response moments is significantly improved.

However, the theory of the proposed technique and the analyses given by the authors has been just demonstrated for non-linear random SDOF systems. Therefore, this paper presents a comprehensive LOMSEC for non-linear random MDOF systems. Thereupon, illustrative examples which include some SDOF and two-degree-of-freedom systems are given for demonstration. For comparison with Caughey's method through the evaluation of accuracy of the solutions, the systems selected for analysis are ones for that exist the known exact solution or the solution acknowledged as exact.

2. Gaussian equivalent linearization

First of all, we recall some basic ideas of the method of GEL. Suppose the mechanical structure discretized by a MDOF system is described by a set of non-linear first order differential equations:

$$\dot{z} = g(z) + f(t), \tag{1}$$

where a dot denotes time differentiation, $z = (z_1, z_2, \ldots, z_n)^{\mathrm{T}}$ is a vector of state variables, n is a natural number, g is a non-linear vector function of components of z, $f(t)$ is a stationary Gaussian random excitation vector, with zero mean. Suppose that a stationary solution to Eq. (1) exists. Denote:

$$e(z) = \dot{z} - g(z) - f(t). \tag{2}$$

Eq. (1) can be rewritten in the form:

$$e(z) = 0. \tag{3}$$

According to the GEL method, we introduce new linear terms in the expression of $e(z)$ as follows:

$$e(z) = \dot{z} - Az + Az - g(z) - f(t), \tag{4}$$

where $A = \{a_{ij}\}$ is a $n \times n$ constant matrix. Let vector y be a stationary solution of the linearized equation:

$$\dot{y} - Ay - f(t) = 0. \tag{5}$$

The vector y is Gaussian since the excitation vector $f(t)$ is Gaussian. Using Eq. (5) one gets from (4):

$$e(y) = Ay - g(y). \tag{6}$$

Thus, if we consider y as an approximation to the solution of the original non-linear equation (1) it is seen that $e(y)$ is an equation error which should be minimized from an optimal criterion. There are some criteria for determining the matrix of linearization, for example, Naess [7], Anh and Schiehlen [9], Socha and Soong [19], etc. The most extensively used criterion is the mean square error criterion, Caughey [1], which requires that the mean squares of error be minimum (here called as Caughey criterion):

$$\langle e_i^2(y) \rangle \to \min_{a_{ij}}, \ i,j = 1, \ldots, n, \tag{7}$$

where $e_i(y)$ are components of $e(y)$. Criterion (7) leads to the necessary condition:

$$A = \langle g(y)y^{\mathrm{T}} \rangle \langle yy^{\mathrm{T}} \rangle^{-1}. \tag{8}$$

From Eq. (8) it is seen that the matrix of linearization A of the linearized equation (5), in turn, depends on the statistics of the response. If in matrix A higher order joint moments of the response appear, they can be expressed in terms of second order moments since y is a Gaussian random vector (see Appendix).

The linear equation (5) can be solved together with Eq. (8) by any of the existing analytical measure using time or frequency domain approaches. Some quite fast cyclic procedures for numerical solutions for GEL may be used, for example, Atalik and Utku [20]:

(a) Assign an initial value to the instantaneous correlation matrix $\langle yy^{\mathrm{T}} \rangle$.
(b) Use Eq. (8) to construct matrix A.
(c) Solve Eq. (5) for the new instantaneous correlation matrix $\langle yy^{\mathrm{T}} \rangle$.
(d) Repeat steps (b) and (c) until results from cycle to cycle are similar.

So, the classical version of GEL as described above, supposes that the minimization of the equation error may give a minimization of the solution error. It should be noted that up to now there is no theoretical proof of GEL; its accuracy has been investigated only by the comparison of the solutions obtained by GEL with their exact solutions if available or with simulation solutions. No mathematical link between the equation error and the solution error has been established. For the full information it should also be noted that there is another version of the mean square error criterion in which the linearized process y in Eq. (8) is replaced by the original non-linear process z. In that version the mean square error criterion can give the exact solution, for example, when the excitation process is white noise one.

3. Local mean square error criterion

3.1. Investigation of the concentrated domain of the response

Denote $p(y)$ the joint probability density function (PDF) of the response vector y to Eq. (5). Criterion (7) can be rewritten in the explicit form:

$$\int_{-\infty}^{+\infty} (n) \int_{-\infty}^{+\infty} e_i^2(y) p(y) \, \mathrm{d}y \to \min_{a_{ij}}. \tag{9}$$

Since the integration is taken over all the co-ordinate space $y \in (-\infty; +\infty)$, criterion (9) may be called as 'global mean square error criterion'. One may propose a concept, which supposes that the global mean square criterion (9) can lead to a large error for some non-linear systems, especially as strong non-linearity. To increase the accuracy, the expected integration should be taken only in a domain where the response vector y is concentrated, Anh and Di Paola [14].

We can bring out the proposed concept through the analysis of some demonstrative examples, which prove that the concentrated domain of the response is narrowed when the non-linearity increases.

3.1.1. Duffing oscillator (non-linear stiffness) under white noise excitation

$$\ddot{x} + 2h\dot{x} + \beta x + \varepsilon x^3 = \sigma w(t). \tag{10}$$

The exact PDF of (10) is known as

$$p(x) = C\exp\left\{-\frac{4h}{\sigma^2}\left(\frac{\beta}{2}x^2 + \frac{\varepsilon}{4}x^4\right)\right\}, \tag{11}$$

where C is the normal constant. Denote Prob$\{-a \leqslant x \leqslant a\}$ as probability for the response dropping in the domain $\{-a, a\}$. If Prob$\{\cdot\}$ is given, then the domain $\{-a, a\}$ will be determined by the following known formula:

$$\text{Prob}\{-a \leqslant x \leqslant a\} = \int_{-a}^{a} p(x)\,\mathrm{d}x. \tag{12}$$

Suppose Prob$\{-a \leqslant x \leqslant a\} = 0.98$, using Eqs. (11) and (12) one gets the value a. Consider the case of the oscillator's parameters as follows: $h = 0.25$; $\sigma = 1$; $\beta = 1$; and ε (non-linearity) varies. The numerical results are given in Table 1. Fig. 1 shows graphically numerical results. It is seen that the response x is concentrated in the limited interval $(-a, a)$ which is narrowed when the non-linearity ε increases.

Table 1
Values a following ε

ε	0.1	0.5	1	5	10	30	50	80	100
a	2.04	1.65	1.46	1.05	0.89	0.69	0.61	0.54	0.51

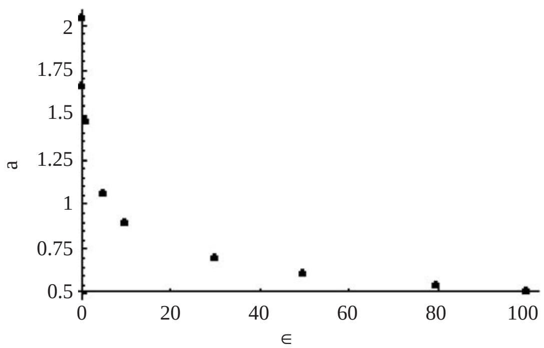


Fig. 1. Values a following ε.

3.1.2. Oscillator with non-linear damping under white noise excitation

$$\ddot{x} + \varepsilon(-1 + x^2 + \dot{x}^2)\dot{x} + x = \sqrt{d}w(t). \tag{13}$$

The exact joint PDF of (13) is known as

$$p(x, \dot{x}) = C \exp\left\{\frac{\varepsilon}{d}\left[\left(x^2 + \dot{x}^2\right) - 0.5\left(x^2 + \dot{x}^2\right)^2\right]\right\}, \tag{14}$$

where C is the normal constant.

For this case, the domain $\{-a, a\}$ can be determined by the following formula:

$$\text{Prob}\{-a \leqslant x \leqslant a\} = \int_{-a}^{a}\left(\int_{-\infty}^{\infty} p(x, \dot{x})\,\mathrm{d}\dot{x}\right)\mathrm{d}x. \tag{15}$$

Analogously, let $\text{Prob}\{-a \leqslant x \leqslant a\} = 0.98$, the corresponding interval $(-a, a)$ are calculated for parameters: $d = 2$ and ε varies. The results are presented in Table 2 and Fig. 2, which give similar observations as in the previous example.

3.1.3. Oscillator with non-linear damping and stiffness under white noise excitation

$$\ddot{x} + 4h\left(\tfrac{1}{2}\dot{x}^2 + \frac{\omega_0^2}{2}x^2 + \frac{\varepsilon}{4}x^4\right)\dot{x} + \omega_0^2 x + \varepsilon x^3 = \sigma w(t). \tag{16}$$

Table 2
Values a following ε

ε	0.1	0.5	1	5	10	30	50	80	100
a	2.92	2.04	1.78	1.36	1.26	1.15	1.11	1.08	1.07

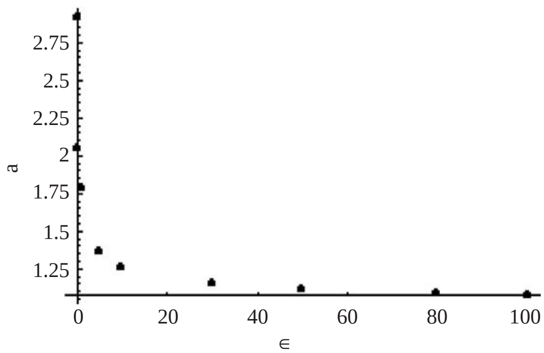


Fig. 2. Values a following ε.

The exact joint PDF of (16) is known as

$$p(x,\dot{x}) = C\exp\left\{-\frac{4h}{\sigma^2}\left(\tfrac{1}{2}\dot{x}^2+\frac{\omega_0^2}{2}x^2+\frac{\varepsilon}{4}x^4\right)^2\right\}, \tag{17}$$

where C is the normal constant.

The domain $\{-a,\ a\}$ is determined by (15) and (17), For $\text{Prob}\{-a \leqslant x \leqslant a\} = 0.98$, $h = 0.1$; $\omega_0^2 = 1$; $\sigma = 1$ and ε varies. The results are presented in Table 3 and Fig. 3.

In all three examples, the obtained results show that the concentrated domain of the response is narrowed when the non-linearity increases. Therefore, for increasing the accuracy of solutions, the expected integration should be taken only in a limited domain where the response is concentrated.

3.2. Local mean square error criterion (LOMSEC)

The concept resulted in the LOMSEC which requires

$$[e_i^2(y)] \to \min_{a_{ij}}, \quad i,j = 1,\ldots,n, \tag{18}$$

where it is denoted

$$[e_i^2(y)] = \int_{-y_1}^{y_1}(n)\int_{-y_n}^{y_n}(\cdot)p(y)\,\mathrm{d}y. \tag{19}$$

To be convenient for using LOMSEC, the integration domains in Eq. (19) should be replaced by non-dimensional ones: $y_n = y_n^0\sigma_{yn}$ (see Appendix), so (19) is replaced by

$$[e_i^2(y)] = \int_{-y_1^0\sigma_{y1}}^{y_1^0\sigma_{y1}}(n)\int_{-y_n^0\sigma_{yn}}^{y_n^0\sigma_{yn}}(\cdot)p(y)\,\mathrm{d}y, \tag{20}$$

Table 3
Values a following ε

ε	0.1	0.5	1	5	10	30	50	80	100
a	1.80	1.51	1.35	0.99	0.85	0.66	0.58	0.51	0.48

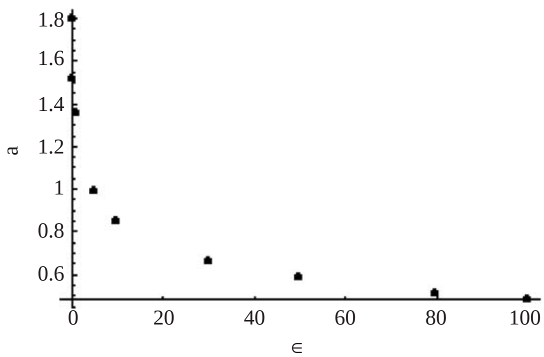


Fig. 3. Values a following ε.

where $y_1^0, y_2^0, \ldots, y_n^0$ are given non-dimension positive values, $\sigma_{y1}, \ldots, \sigma_{yn}$ are square roots of variances of components $y_1, y_2, \ldots, y_n$. It is noted that as in GEL the values $\sigma_{y1}, \ldots, \sigma_{yn}$ are considered as independent parameters from a_{ij} when minimizing (18). Thus, LOMSEC (18) yields the necessary conditions similar to (8):

$$A = [g(y)y^{\mathrm{T}}]\,[yy^{\mathrm{T}}]^{-1}. \tag{21}$$

A cyclic procedure may also be obtained as follows:

(a) Give positive value y_1^0, y_2^0, $\ldots, y_n^0$.
(b) Assign an initial value to the instantaneous correlation matrix $[yy^{\mathrm{T}}]$.
(c) Use (21) to construct matrix A.
(d) Solve (5) for the new instantaneous correlation matrix $[yy^{\mathrm{T}}]$.
(e) Repeat steps (c) and (d) until results from cycle to cycle are similar.

The criterion (20) proves that by the way of changing the limitation of integration domain, the LOMSEC provides with a series of different approximate solutions, and as $y_n^0 = \infty$ LOMSEC gives Caughey solution.

The given concept for the proposed criterion implies existence of the optimal values of y_n^0 for a specific system, which allow getting as the best approximate solution as possible. However, it is impossible so far to establish the mathematical link between such values of y_n^0 and value of the system parameters, especially the non-linearity parameter. This is a significant limitation of the criterion. To deal with this obstacle, a measure realized in Section 4 as below.

For MDOF stochastic non-linear systems, LOMSEC only gives the ultimate result in the form of numerical, because the analytic calculation is very difficult. Thus, if necessary to obtain higher accurate result, some numerical methods should be directly applied, for example Runge–Kutta.

4. Analysis of second moments of response

This section will compare the approximate solutions obtained by using Caughey criterion and the proposed technique for representative non-linear systems under Gaussian white noise random excitation. However, LOMSEC solution depends on $y^0 = (y_1^0, y_2^0, \ldots, y_n^0)$, so a question appeared is that how to choose y^0 in order to obtain as most improved solution as possible. The idea for study is as follows:

- Based on the systems for that exist exact solutions, try to find y_e^0 corresponding to the exact solution. We can gain this purpose through solving inverse problem of which the unknown response moment in the equation for LOMSEC solution is replaced by the known exact one. There are values y_e^0 corresponding to various non-linearity of the system.
- Choose an averaging value y_a^0 for obtaining rationally accurate solution at any non-linearity with concern over strong and mean non-linearity. The y_a^0 can be calculated as follows: $y_a^0 = [(y_e^0max + y_e^0min)/2]$, where y_e^0max and y_e^0min are the largest and smallest, respectively, from the sequence of values y_e^0 corresponding to various non-linearity of the system.
- Through a series of systems analyzed, we can recommend a y^0 for application to any non-linear system. By this way, the applicability of LOMSEC can approach to reality.

Example 1. *Duffing oscillator (non-linear stiffness):* Consider Duffing oscillator (10), where $w(t)$ is white noise excitation with unit intensity. The exact second moments can be directly found from PDF (11)

$$\langle x^2 \rangle_e = \frac{\int_0^{+\infty} x^2 \mathrm{Exp}\{(-4h/\sigma^2)((\beta/2)x^2 + (\varepsilon/4)x^4)\}\,\mathrm{d}x}{\int_0^{+\infty} \mathrm{Exp}\{(-4h/\sigma^2)((\beta/2)x^2 + (\varepsilon/4)x^4)\}\,\mathrm{d}x}. \tag{22}$$

Suppose $\varepsilon x^3 = \lambda x$, the linearized equation corresponding to (10) is governed by

$$\ddot{x} + 2h\dot{x} + (\beta + \lambda)x = \sigma w(t). \tag{23}$$

The solution of the linearized equation (23) is to be

$$\langle x^2 \rangle = \frac{\sigma^2}{4h(\beta + \lambda)}. \tag{24}$$

The coefficient of linearization λ is determined by different criteria.

Using Caughey criterion one gets

$$\lambda = \varepsilon \frac{\langle x^4 \rangle}{\langle x^2 \rangle} = 3\varepsilon \langle x^2 \rangle. \tag{25}$$

Put (25) in (24) and denote $\langle x \rangle_G$ as the solution obtained by Caughey criterion:

$$\langle x^2 \rangle_G = \frac{-\beta + \sqrt{\beta^2 + 3\varepsilon\sigma^2/h}}{6\varepsilon}. \tag{26}$$

For LOMSEC one gets

$$\lambda = \varepsilon \frac{[x^4]_{-x^0}^{x^0}}{[x^2]_{-x^0}^{x^0}} = K_{x^0}\varepsilon \langle x^2 \rangle, \tag{27}$$

where (see Appendix)

$$K_{x^0} = \frac{\int_0^{x^0} t^4 n(t)\,\mathrm{d}t}{\int_0^{x^0} t^2 n(t)\,\mathrm{d}t}. \tag{28}$$

Put (27) in (24) and denote $\langle x \rangle_{LG}$ as the solution obtained by LOMSEC criterion:

$$\langle x^2 \rangle_{LG} = \frac{-\beta + \sqrt{\beta^2 + K_{x^0}\varepsilon\sigma^2/h}}{2K_{x^0}\varepsilon}. \tag{29}$$

Consider the case $\beta = 1$; $h = 0.25$; $\sigma = 1$; meanwhile ε varies. Use (22) and (29) for finding x_e^0 corresponding to the exact second moments. The results are given in Table 4. Fig. 4 shows the dependence of LOMSEC solution on the value x^0, for $\varepsilon = 100$. The averaging value x_a^0 for LOMSEC (29) calculated as $(2.95488 + 2.32620)/2 = 2.64054$, approximately $x_a^0 = 2.6$. The numerical results are presented in Table 5 where E_G and E_{LG} denote the errors of $\langle x^2 \rangle_G$ and $\langle x^2 \rangle_{LG}$ versus $\langle x^2 \rangle_e$, respectively.

Table 4
Values x_e^0 depending on the non-linearity

ε	0.1	1	10	100
$\langle x^2 \rangle_e$	0.81756	0.46792	0.18890	0.06496
x_e^0	2.95488	2.54211	2.38034	2.32620

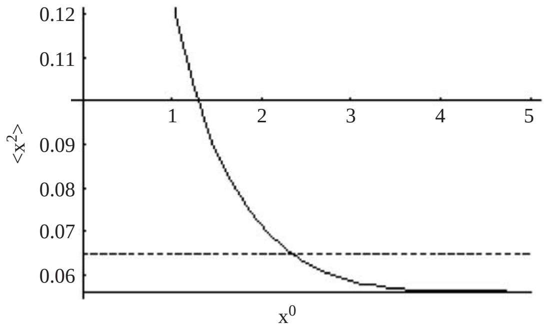


Fig. 4. Dependence of LOMSEC solution on x^0, for $\varepsilon = 100$. Denote: ---- exact, — Caughey, (LOMSEC.

Table 5
Second moment of response

ε	0.1	1	10	100
$\langle x^2 \rangle_G (E_G\%)$	0.80540 (−1.487)	0.43426 (−7.194)	0.16667 (−11.768)	0.05609 (−13.655)
$\langle x^2 \rangle_{LG} (E_{LG}\%)$	0.82935 (1.442)	0.46456 (−0.718)	0.18162 (−3.854)	0.06150 (−5.326)

Table 6
Values x_e^0 depending on the non-linearity

ε	0.1	1	10	100
$\langle x^2 \rangle_e$	8.71363	1.04180	0.24352	0.07039
x_e^0	1.58223	2.04041	2.21955	2.27553

Another case is also considered namely, for $\beta = -1$; $h = 0.25$; $\sigma = 1$; ε varies. It is seen that the value x_e^0 corresponding to the exact moment is reduced even to 1.58 (for the case $\varepsilon = 0.1$). For that reason, the averaging value x_a^0 is proposed as $(2.04041 + 2.27553)/2 = 2.15797$, approximately $x_a^0 = 2.2$. The results for this case are presented in Tables 6 and 7 and Fig. 5.

Table 7
Second moment of response

ε	0.1	1	10	100
$\langle x^2 \rangle_G$ (E_G%)	4.13873 (−52.503)	0.76759 (−26.321)	0.20000 (−17.871)	0.05943 (−15.570)
$\langle x^2 \rangle_{LG}$ (E_{LG}%)	5.67100 (−34.918)	0.97602 (−6.314)	0.24499 (0.604)	0.07189 (2.131)

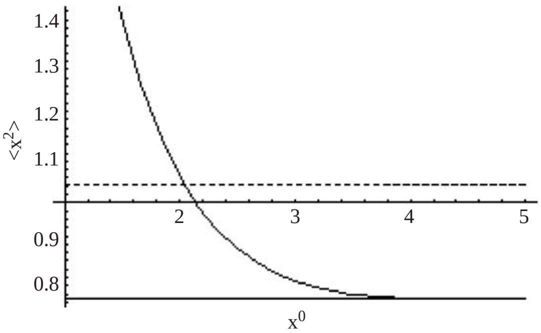


Fig. 5. Dependence of LOMSEC solution on x^0, for $\varepsilon = 1$. Denote: ---- exact, — Caughey, ⎝ LOMSEC.

Example 2. *Oscillator with non-linear damping following velocity*:

$$\ddot{x} + 2\varepsilon\dot{x} + 2\varepsilon\gamma\dot{x}^3 + \omega_0^2 x = \sigma w(t), \tag{30}$$

where $w(t)$ is white noise excitation with unit intensity. The exact solution of (30) has not been found yet. The solution found by the equivalent non-linearization $\langle x^2 \rangle_{ENLE}$ is regarded as the exact solution, Roberts and Spanos [5]. The equivalent linearized system corresponding to (30) is

$$\ddot{x} + (2\varepsilon + \mu)\dot{x} + \omega_0^2 x = \sigma w(t). \tag{31}$$

The solution of the linearized equation (31) is

$$\langle x^2 \rangle = \frac{\sigma^2}{2(2\varepsilon + \mu)\omega_0^2}. \tag{32}$$

Caughey criterion resulted in below

$$\mu = 2\varepsilon\gamma \frac{\langle \dot{x}^4 \rangle}{\langle \dot{x}^2 \rangle} = 6\varepsilon\gamma \langle \dot{x}^2 \rangle. \tag{33}$$

Combining (32) and (33) and using $\langle \dot{x}^2 \rangle = \langle x^2 \rangle \omega_0^2$ one gets the Caughey solution:

$$\langle x^2 \rangle_G = \frac{\sigma^2}{4\varepsilon(1 + 3\gamma\langle x^2 \rangle \omega_0^2)\omega_0^2}. \tag{34}$$

LOMSEC criterion resulted in below

$$\mu = 2K_{x^0}\varepsilon\gamma \langle \dot{x}^2 \rangle, \tag{35}$$

where

$$K_{x^0} = \frac{\int_0^{x^0} t^4 n(t)\,\mathrm{d}t}{\int_0^{x^0} t^2 n(t)\,\mathrm{d}t}. \tag{36}$$

Combining (32) and (35) one gets the LOMSEC solution as follows:

$$\langle x^2 \rangle_{LG} = \frac{\sigma^2}{4\varepsilon(1 + \gamma\langle \dot{x}^2 \rangle K_{x^0})\omega_0^2}. \tag{37}$$

Consider the case $\sigma = \sqrt{4\varepsilon}$; $\omega_0^2 = 1$; $\varepsilon = 0.05$; and let γ varies. The averaging value x_a^0 is calculated as $x_a^0 = 2.6$. The numerical results are presented in Tables 8 and 9 and Fig. 6. It is seen that the approximate solution obtained by LOMSEC is better than that by Caughey.

Table 8
Values x_e^0 depending on the non-linearity

γ	1	3	5	8	10
$\langle x^2 \rangle_{ENLE}$	0.4603	0.3058	0.2476	0.2025	0.1835
x_e^0	2.65	2.59	2.58	2.56	2.55

Table 9
Second moment of response

γ	1	3	5	8	10
$\langle x^2 \rangle_G$ (E_G%)	0.4342 (−5.7)	0.2824 (−7.6)	0.2270 (−8.3)	0.1843 (−9.0)	0.1667 (−9.1)
$\langle x^2 \rangle_{LG}$ (E_{LG}%)	0.4633 (0.65)	0.3045 (−0.42)	0.2456 (−0.81)	0.2000 (−1.23)	0.1810 (−1.36)

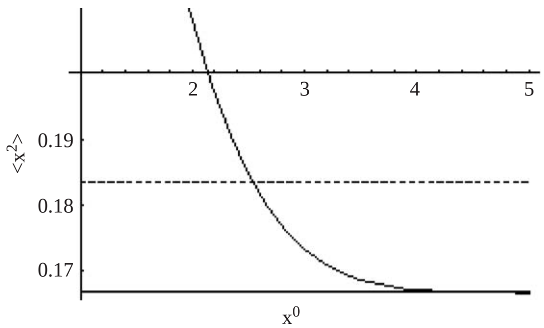


Fig. 6. Dependence of LOMSEC solution on x^0, for $\gamma = 10$. Denote: ---- exact, — Caughey, ⎝ LOMSEC.

Example 3. *Oscillator with non-linear stiffness and damping:* Consider oscillator with non-linear stiffness and damping (16), where $w(t)$ is white noise excitation with unit intensity. The PDF (17) resulted in the following exact solution of (16):

$$\langle x^2 \rangle_e = \frac{\int_0^{+\infty}\int_0^{+\infty} x^2 \,\mathrm{Exp}\left\{(-4h/\sigma^2)\left(\frac{1}{2}\dot{x}^2 + (\omega_0^2/2)x^2 + (\varepsilon/4)x^4\right)^2\right\} \mathrm{d}x\,\mathrm{d}\dot{x}}{\int_0^{+\infty}\int_0^{+\infty} \mathrm{Exp}\left\{(-4h/\sigma^2)\left(\frac{1}{2}\dot{x}^2 + (\omega_0^2/2)x^2 + (\varepsilon/4)x^4\right)^2\right\} \mathrm{d}x\,\mathrm{d}\dot{x}}. \tag{38}$$

Let $4h(\frac{1}{2}\dot{x}^2 + (\omega_0^2/2)x^2 + (\varepsilon/4)x^4)\dot{x} = \mu\dot{x}$, and $\varepsilon x^3 = \lambda x$.

The equivalent linearized system corresponding to (16) is governed by

$$\ddot{x} + \mu\dot{x} + (\omega_0^2 + \lambda)x = \sigma w(t). \tag{39}$$

The solution of the linearized equation (39) is known as

$$\langle x^2 \rangle = \frac{\sigma^2}{2\mu(\omega_0^2 + \lambda)}. \tag{40}$$

Caughey criterion resulted in the following linearization coefficients:

$$\lambda = 3\varepsilon\langle x^2 \rangle \quad \text{and} \quad \mu = 2h\left(3\langle \dot{x}^2 \rangle + \omega_0^2\langle x^2 \rangle + \frac{3\varepsilon}{2}\langle x^2 \rangle^2\right). \tag{41}$$

It is proved that $\langle \dot{x}^2 \rangle = \langle x^2 \rangle(\omega_0^2 + \lambda)$, so the linearization coefficient μ in Eq. (41) is to be

$$\mu = h(8\omega_0^2\langle x^2 \rangle + 21\varepsilon\langle x^2 \rangle^2). \tag{42}$$

Caughey solution is obtained from (40), (41) and (42)

$$63\varepsilon^2\langle x^2 \rangle_G^4 + 45\varepsilon\omega_0^2\langle x^2 \rangle_G^3 + 8\omega_0^4\langle x^2 \rangle_G^2 - \frac{\sigma^2}{2h} = 0. \tag{43}$$

By the similar steps one gets the following linearization coefficients for LOMSEC

$$\lambda = K_{x^0}\varepsilon\langle x^2 \rangle \text{ and } \mu = h\{2(K_{x^0} + H_{x^0})\omega_0^2\langle x^2 \rangle + \varepsilon(2K_{x^0}^2 + I_{x^0})\langle x^2 \rangle^2\}, \tag{44}$$

where

$$K_{x^0} = \frac{\int_0^{x^0} t^4 n(t)\,\mathrm{d}t}{\int_0^{x^0} t^2 n(t)\,\mathrm{d}t}, \quad H_{x^0} = \frac{\int_0^{x^0} t^2 n(t)\,\mathrm{d}t}{\int_0^{x^0} n(t)\,\mathrm{d}t}, \quad I_{x^0} = \frac{\int_0^{x^0} t^4 n(t)\,\mathrm{d}t}{\int_0^{x^0} n(t)\,\mathrm{d}t}. \tag{45}$$

LOMSEC solution is obtained from (40) and (44)

$$\varepsilon^2 K_{x^0}(2K_{x^0}^2 + I_{x^0})\langle x^2 \rangle_{LG}^4 + \varepsilon\omega_0^2(4K_{x^0}^2 + 2K_{x^0}H_{x^0} + I_{x^0})\langle x^2 \rangle_{LG}^3 + 2\omega_0^4(K_{x^0} + H_{x^0})\langle x^2 \rangle_{LG}^2 - \frac{\sigma^2}{2h} = 0. \tag{46}$$

Suppose $\omega_0^2 = 1$, $\sigma^2 = 1$, $h = 0.1$; ε varies. The results are presented in Tables 10 and 11 and Fig. 7. For LOMSEC solution, the averaging value x_a^0 is calculated as $x_a^0 = 2.3$.

Table 10
Values x_e^0 depending on the non-linearity

ε	0.1	1	10	100
$\langle x^2 \rangle_e$	0.76773	0.46521	0.19241	0.06663
x_e^0	2.36895	2.30081	2.23646	2.21042

Table 11
Second moment of response

ε	0.1	1	10	100
$\langle x^2 \rangle_G$ (E_G%)	0.66590 (−13.264)	0.38157 (−17.979)	0.15094 (−21.553)	0.05135 (−22.933)
$\langle x^2 \rangle_{LG}$ (E_{LG}%)	0.78087 (1.712)	0.46533 (0.026)	0.18802 (−2.282)	0.06440 (−3.347)

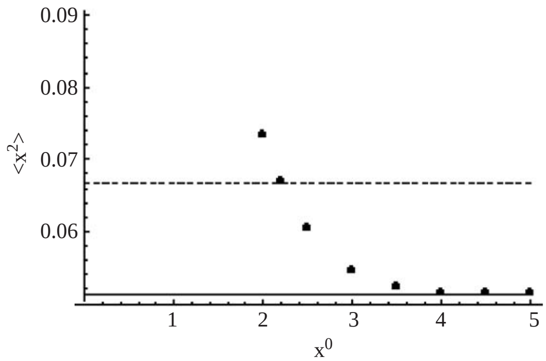


Fig. 7. Dependence of LOMSEC solution on x^0, for $\varepsilon = 100$. Denote: ---- exact, — Caughey, ● ● ● LOMSEC.

Example 4. *Two-degree-of-freedom oscillator with non-linear stiffness:* Consider the following non-linear random two-degree-of-freedom system, which was analyzed by Wen Yao Jia and Tong Fang using an approximate PDF method [21]:

$$\ddot{x}_i + \beta_i \dot{x}_i + \frac{\partial}{\partial x_i} U(x_1, x_2) = w_i(t), \quad i = 1, 2, \tag{47}$$

where

$$U(x_1, x_2) = \tfrac{1}{2}\omega_1^2 x_1^2 + \tfrac{1}{2}\omega_2^2 x_2^2 + \lambda_1 x_1^4 + \lambda_3 x_1^2 x_2^2 + \lambda_5 x_2^4. \tag{48}$$

Under the following assumptions:

$$\langle w_i(t) \rangle = 0 \, (i = 1, 2); \quad \langle w_i(t) w_j(t+\tau) \rangle = 2\pi k_i \delta_{ij} \delta(\tau) \quad (i, j = 1, 2);$$
$$\beta_i = R k_i \quad (i = 1, 2). \tag{49}$$

The corresponding Fokker–Planck equation has an exact solution for the stationary PDF:

$$f(x_1, x_2) = C\exp\left\{-\frac{R}{\pi}U(x_1, x_2)\right\}, \tag{50}$$

where C is determined by the normalization condition. The exact solutions of (47) are

$$\langle x_i^2 \rangle_e = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_i^2 f(x_1, x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2, \quad i = 1, 2 \tag{51}$$

using (48) original (47) system can be rewritten

$$\begin{aligned} &\ddot{x}_1 + \beta_1\dot{x}_1 + \omega_1^2 x_1 + 4\lambda_1 x_1^3 + 2\lambda_3 x_1 x_2^2 = w_1(t),\\ &\ddot{x}_2 + \beta_2\dot{x}_2 + \omega_2^2 x_2 + 4\lambda_5 x_2^3 + 2\lambda_3 x_1^2 x_2 = w_2(t). \end{aligned} \tag{52}$$

Since the whole linear part in each equation of system (52) only contains an independent variable, the analytical procedure can be simply conducted similar to the procedure for SDOF systems by the following substitutes:

$$\begin{aligned} &4\lambda_1 x_1^3 + 2\lambda_3 x_1 x_2^2 = \rho_1 x_1,\\ &4\lambda_5 x_2^3 + 2\lambda_3 x_1^2 x_2 = \rho_2 x_2. \end{aligned} \tag{53}$$

The linearized system is governed by the following two-equation system:

$$\ddot{x}_i + \beta_i\dot{x}_i + (\omega_i^2 + \rho_i)x_i = w_i(t), \quad i = 1, 2. \tag{54}$$

The assumption (53) leads to an equation error as follows:

$$\begin{aligned} &e_1 = 4\lambda_1 x_1^3 + 2\lambda_3 x_1 x_2^2 - \rho_1 x_1,\\ &e_2 = 4\lambda_5 x_2^3 + 2\lambda_3 x_1^2 x_2 - \rho_2 x_2. \end{aligned} \tag{55}$$

The solution of linearized system (54) is found:

$$\langle x_i^2 \rangle = \frac{\pi}{R(\omega_i^2 + \rho_i)}, \quad i = 1, 2. \tag{56}$$

Caughey criterion yields:

$$\left\langle e_1\frac{\partial e_1}{\partial \rho_1}\right\rangle = 0, \quad \left\langle e_2\frac{\partial e_2}{\partial \rho_2}\right\rangle = 0. \tag{57}$$

Expanding condition (57) and in combination with (55)–(56) we get a closure-equation system that leads to the Caughey solution:

$$\begin{aligned} &12\lambda_1\langle x_1^2\rangle_G^2 + 2\lambda_3\langle x_1^2\rangle_G\langle x_2^2\rangle_G + \omega_1^2\langle x_1^2\rangle_G - \frac{\pi}{R} = 0,\\ &12\lambda_5\langle x_2^2\rangle_G^2 + 2\lambda_3\langle x_1^2\rangle_G\langle x_2^2\rangle_G + \omega_2^2\langle x_2^2\rangle_G - \frac{\pi}{R} = 0. \end{aligned} \tag{58}$$

LOMSEC yields:

$$\left[e_1\frac{\partial e_1}{\partial \rho_1}\right]_{-x_1^0\sigma_{x1}}^{x_1^0\sigma_{x1}} = 0, \quad \left[e_2\frac{\partial e_2}{\partial \rho_2}\right]_{-x_2^0\sigma_{x2}}^{x_2^0\sigma_{x2}} = 0, \tag{59}$$

Table 12
Values x_{1e}^0, x_{2e}^0 depending on the non-linearity

μ	0.1	1	10	100
$\langle x_1^2 \rangle_e$	1.17821	0.60378	0.22519	0.07462
x_{1e}^0	2.86613	2.53319	2.42279	2.39088
$\langle x_2^2 \rangle_e$	0.37680	0.30640	0.16987	0.06766
x_{2e}^0	3.45000	2.94430	2.59630	2.44672

Table 13
Second moments of responses x_1 and x_2

μ	0.1	1	10	100
$\langle x_1^2 \rangle_G$ (E_{G1}%)	1.15140	0.55671	0.20077	0.06591
	(−2.275)	(−7.796)	(−10.844)	(−11.673)
$\langle x_1^2 \rangle_{LG}$ (E_{LG1}%)	1.18163	0.58453	0.21244	0.06986
	(0.290)	(−3.188)	(−5.662)	(−6.379)
$\langle x_2^2 \rangle_G$ (E_{G2}%)	0.37664	0.30284	0.16015	0.06122
	(−0.042)	(−1.162)	(−5.722)	(−9.518)
$\langle x_2^2 \rangle_{LG}$ (E_{LG2}%)	0.37796	0.30849	0.16690	0.06454
	(0.308)	(0.682)	(−1.748)	(−4.611)

where $(-x_i^0\sigma_{xi}, x_i^0\sigma_{xi})$ are the expected integration domains, σ_{xi} are square roots of variances of components $x_i (i = 1, 2)$, x_i^0 is a given positive values. Expanding condition (59) in combination with (55)–(56), we get a closure-equation system leading to LOMSEC solution:

$$4\lambda_1 K_{x_1^0}\langle x_1^2\rangle_{LG}^2 + 2\lambda_3 H_{x_2^0}\langle x_1^2\rangle_{LG}\langle x_2^2\rangle_{LG} + \omega_1^2\langle x_1^2\rangle_{LG} - \frac{\pi}{R} = 0,$$
$$4\lambda_5 K_{x_2^0}\langle x_2^2\rangle_{LG}^2 + 2\lambda_3 H_{x_1^0}\langle x_1^2\rangle_{LG}\langle x_2^2\rangle_{LG} + \omega_2^2\langle x_2^2\rangle_{LG} - \frac{\pi}{R} = 0, \quad (60)$$

where

$$K_{x_i^0} = \frac{\int_0^{x_i^0} t^4 n(t)\,dt}{\int_0^{x_i^0} t^2 n(t)\,dt}, \quad H_{x_i^0} = \frac{\int_0^{x_i^0} t^2 n(t)\,dt}{\int_0^{x_i^0} n(t)\,dt}, \quad i = 1, 2. \quad (61)$$

Consider the case: $R = 0.5$; $\beta_1 = \beta_2 = 0.1$; $\omega_1 = 2$; $\omega_2 = 4$; $\lambda_1 = \lambda_3 = \lambda_5 = \mu$ (varies). The numerical results are presented in Tables 12 and 13, Figs. 8 and 9. For this system, if we choose the averaging values $x_{1a}^0 \neq x_{2a}^0$ for calculating response moments $\langle x_1^2\rangle_{LG}$ and $\langle x_2^2\rangle_{LG}$, respectively, i.e., $x_{1a}^0 = (2.86613 + 2.39088)/2 = 2.62851$ and $x_{2a}^0 = (3.45000 + 2.44672)/2 = 2.94836$, the accuracy will be better for each response moment. Nevertheless, for simplification, here we choose $x_{1a}^0 = x_{2a}^0 = 2.8$ resulted approximately from $(2.62851 + 2.94836)/2$.

In considered systems, the exact solutions are always bigger than Caughey's ones, meanwhile LOMSEC solutions vary in accordance with values of the integration domain (x_i^0). In Figs. 4–9 we can see that the curve of LOMSEC solution always crosses the exact solution at a point in accordance with a defined value of x_e^0 and approaches to Caughey one when $x_i^0 \to \infty$. This proves the existence of a value x_e^0 in LOMSEC that allows obtaining the exact solution.

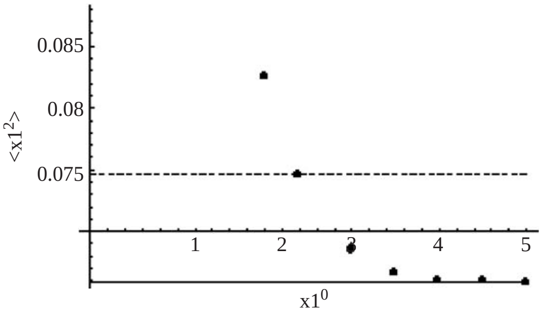


Fig. 8. Dependence of LOMSEC solution on x_1^0, for $\mu = 100$. Denote: ---- exact, — Caughey, ● ● ● LOMSEC.

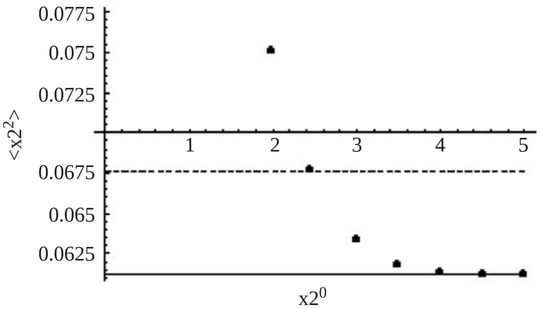


Fig. 9. Dependence of LOMSEC solution on x_2^0, for $\mu = 100$. Denote: ---- exact, — Caughey, ● ● ● LOMSEC.

By the way of choosing the approximate value x_a^0 for LOMSEC solution as the above presented, we gain the accuracy of LOMSEC solution much more improved than that of Caughey, especially as strong non-linearity. This is the most significant advantage of the proposed criterion. In a determined non-linearity coverage there exists a point where the error of LOMSEC solution is zero (the error changes the sign), meanwhile this does not happen for Caughey solution. However, LOMSEC solution for some cases of weak non-linearity may be worse than that by Caughey (see Example 4: $E_{LG2} > E_{G2}$ as $\mu = 0.1$).

As mentioned in Section 3 that the mathematical link between the expected integration and the system parameters cannot be found so far. This is a key limitation of the proposed method. In such situation, an acceptable measure is that to recommend an approximate integration interval for practical application drawn from the numerical investigation of a series of various non-linear systems whose exact or simulated solutions found (analogously as the above examples).

5. Analysis of the mean up-crossing rate

In the analysis of non-linear random vibrations, besides the second order moment, the PDF and the mean up-crossing rate (MCR) of the system responses are some quantities of main concern. The fact that the analysis of the second order moments using the equivalent linearization technique were very much investigated whereas the researches on MCR in the linearization were rarely made. MCR frequently is used in estimating the reliability or the exceedance probability of response [7,18,22].

The mean up-crossing rate is the rate at which a differentiable stationary response process $X(t)$ crosses a level $X = x$ with a positive slope. The evaluation of MCR is based on the PDF of the responses of randomly excited systems as follows:

$$v_e(x) = \int_0^{\infty} \dot{x} p_e(x, \dot{x})\, \mathrm{d}\dot{x}, \tag{62}$$

where $p_e(x, \dot{x})$ is the exact joint PDF of responses $x(t)$ and $\dot{x}(t)$, $\dot{x} = \mathrm{d}x/\mathrm{d}t$. So, v_e is refereed to as the exact mean up-crossing rate (EMCR). However, it is difficult to obtain the exact PDF of randomly excited non-linear systems in practice. Even if the response can be modelled as a Markov process, the possibility of an exact PDF solution is still much limited. Therefore, some approximate methods were developed and investigated for estimating the approximate mean up-crossing rate (AMCR) (see, e.g., Refs. [23,24]).

This section presents the analysis of AMCR using Caughey criterion and LOMSEC. Two systems chosen for analysis here are taken out from the representative systems presented in the previous section. Comparisons of AMCRs versus EMCR are also given. It is obvious that Eq. (62) is EMCR of the original non-linear system, so the equivalent linearization method resulted in AMCR which can be governed by

$$v_A(x) = \int_0^{\infty} \dot{x} p_A(x, \dot{x})\, \mathrm{d}\dot{x}, \tag{63}$$

where p_A is the approximate joint PDF obtained from the equivalent linearized system and considered as the normal

$$p_A(x, \dot{x}) = \frac{1}{2\pi\sigma_x\sigma_{\dot{x}}} \exp\left\{ -\left(\frac{x^2}{2\sigma_x^2} + \frac{\dot{x}^2}{2\sigma_{\dot{x}}^2} \right) \right\}, \tag{64}$$

where $\sigma_x^2 = \langle x^2 \rangle, \sigma_{\dot{x}}^2 = \langle \dot{x}^2 \rangle$ are the second moments obtained from linearized systems.

Example 5. Consider again Duffing oscillator (10). The exact joint PDF (10) is known as

$$p_e(x, \dot{x}) = \frac{\sqrt{h} \exp\left\{ -\frac{2h\dot{x}^2}{\sigma^2} \right\} \exp\left\{ -\frac{4h}{\sigma^2}\left(\frac{\beta}{2}x^2 + \frac{\varepsilon}{4}x^4 \right) \right\}}{\sigma\sqrt{2\pi} \int_0^{\infty} \exp\left\{ -\frac{4h}{\sigma^2}\left(\frac{\beta}{2}x^2 + \frac{\varepsilon}{4}x^4 \right) \right\} \mathrm{d}x}. \tag{65}$$

Using (62) one gets the EMCR as follows:

$$v_e(x) = \frac{\sqrt{h}\left(\int_0^{\infty} \dot{x} \exp\left\{ -\frac{2h\dot{x}^2}{\sigma^2} \right\} \mathrm{d}\dot{x} \right) \exp\left\{ -\frac{4h}{\sigma^2}\left(\frac{\beta}{2}x^2 + \frac{\varepsilon}{4}x^4 \right) \right\}}{\sigma\sqrt{2\pi} \int_0^{\infty} \exp\left\{ -\frac{4h}{\sigma^2}\left(\frac{\beta}{2}x^2 + \frac{\varepsilon}{4}x^4 \right) \right\} \mathrm{d}x}. \tag{66}$$

From (63) to (64) and with attention that $\langle \dot{x}^2 \rangle = \langle x^2 \rangle(\beta + \lambda)$, one gets AMCR

$$v_A(x) = \frac{\left(\int_0^\infty \dot{x} \exp\{-\dot{x}^2/2\langle x^2 \rangle(\beta+\lambda)\}\, \mathrm{d}\dot{x}\right)\exp\{-x^2/2\langle x^2 \rangle\}}{2\pi\langle x^2 \rangle\sqrt{\beta+\lambda}}, \tag{67}$$

where λ, $\langle x^2 \rangle$ are the linearization coefficient and the second order moment, respectively. Denote λ_G, $\langle x^2 \rangle_G$ obtained by Caughey's and λ_{LG}, $\langle x^2 \rangle_{LG}$ by LOMSEC, one gets $v_G(x)$ and $v_{LG}(x)$, respectively. The values of λ and $\langle x^2 \rangle$ are known in Example 1.

Consider the case $\beta = -1$; $h = 0.25$; $\sigma = 1$; ε varies. The expressions for estimation of MCR are given in Table 14, which are the results obtained from (67) to (68). The graphically numerical results are shown in Figs. 10a and b for $\varepsilon = 1$ and Figs. 11a and b for $\varepsilon = 100$. Fig. 10b

Table 14
Expressions for estimation of MCR

MCR	$\varepsilon = 1$	$\varepsilon = 100$
$v_e(x)$	$0.10216 \exp\{0.5x^2 - 0.25x^4\}$	$0.47541 \exp\{0.5x^2 - 25x^4\}$
$v_G(x)$	$0.18166 \exp\{-0.65139x^2\}$	$0.65290 \exp\{-8.41326x^2\}$
$v_{LG}(x)$	$0.16110 \exp\{-0.51229x^2\}$	$0.59363 \exp\{-6.95507x^2\}$

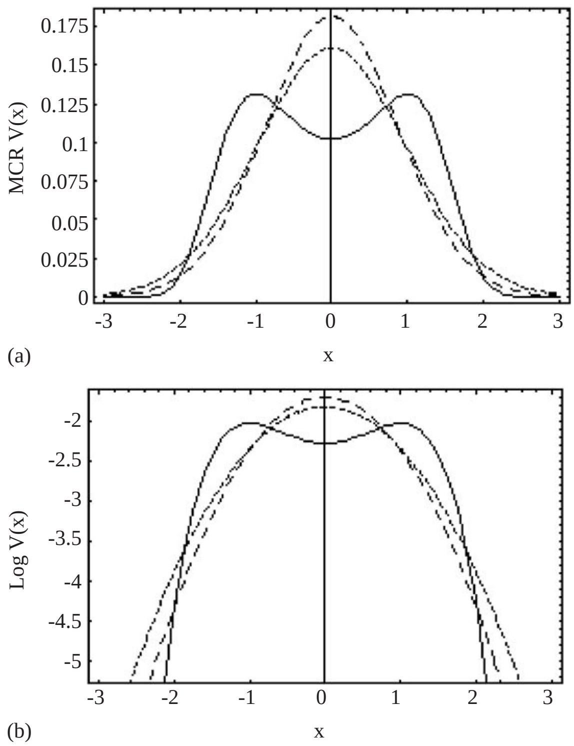


Fig. 10. (a) MCR, for $\varepsilon = 1$. Denote: — v_e, – – v_G, - - - - v_{LG}. (b) Logarithmic of MCR, for $\varepsilon = 1$. Denote: — v_e, – – v_G, - - - - v_{LG}.

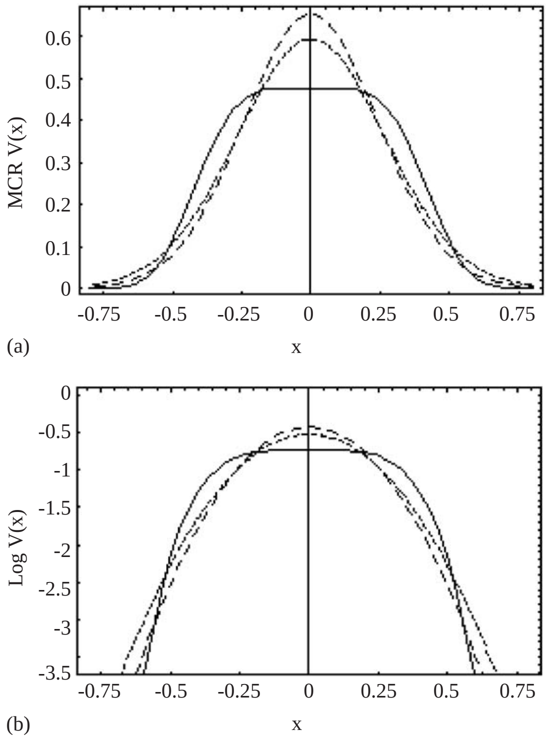


Fig. 11. (a) MCR, for $\varepsilon = 100$. Denote: — v_e, – – v_G, - - - - v_{LG}. (b) Logarithmic of MCR, for $\varepsilon = 100$. Denote: — v_e, – – v_G, - - - - v_{LG}.

and Fig. 11b show the logarithmic of MCRs in order to compare the tail behavior of AMCRs to EMCR.

Example 6. Consider again the oscillator with non-linear stiffness and damping (16). The oscillator has the joint PDF known as (17). By the same steps one gets

$$v_e(x) = \frac{\int_0^\infty \dot{x} \exp\left\{-\frac{4h}{\sigma^2}\left(\frac{1}{2}\dot{x}^2 + \frac{\omega_0^2}{2}x^2 + \frac{\varepsilon}{4}x^4\right)^2\right\} \mathrm{d}\dot{x}}{4\int_0^\infty \int_0^\infty \exp\left\{-\frac{4h}{\sigma^2}\left(\frac{1}{2}\dot{x}^2 + \frac{\omega_0^2}{2}x^2 + \frac{\varepsilon}{4}x^4\right)^2\right\} \mathrm{d}x\, \mathrm{d}\dot{x}}. \tag{68}$$

The equation for estimation of AMCR is the same as (67) in which β is replaced by ω_0^2. Consider the case $\omega_0^2 = 1$, $\sigma = 1$, $h = 0.25$ and ε varies. The expressions for estimation of MCR are given in Table 15. The graphically numerical results are shown in Figs. 12a and b and Figs. 13a and b.

Through the above-considered examples, it is seen that for small level x the solution v_{LG} is more closed to the exact solution v_e than v_G is. The small level x is narrowed when the non-linearity increases. For larger level x, v_G and v_{LG} are both generally not good at any non-linearity.

Table 15
Expressions for estimation of MCR

MCR	$\varepsilon = 1$	$\varepsilon = 100$
$v_e(x)$	$0.21975 \int_0^\infty \dot{x} \exp\{-(0.5\dot{x}^2 + 0.5x^2 + 0.25x^4)^2\} d\dot{x}$	$0.52733 \int_0^\infty \dot{x} \exp\{-(0.5\dot{x}^2 + 0.5x^2 + 25x^4)^2\} d\dot{x}$
$v_G(x)$	$0.21587 \exp\{-1.78642x^2\}$	$0.57688 \exp\{-12.35790x^2\}$
$v_{LG}(x)$	$0.21003 \exp\{-1.47501x^2\}$	$0.55349 \exp\{-9.85804x^2\}$

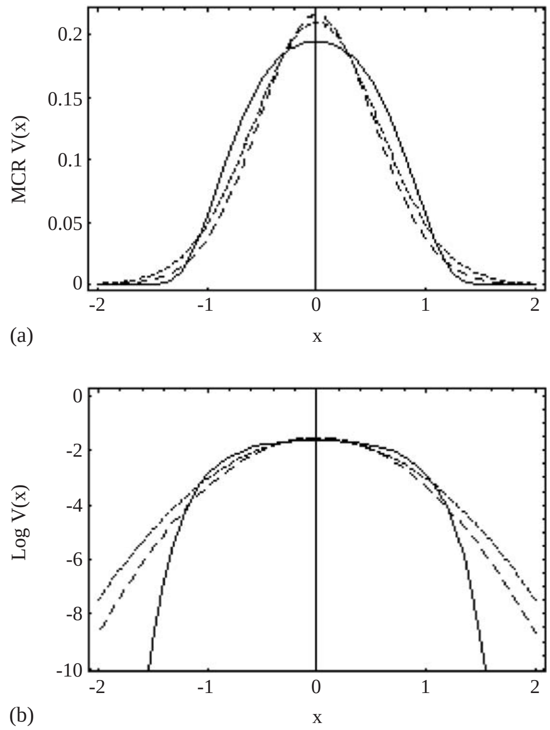


Fig. 12. (a) MCR, for $\varepsilon = 1$. Denote: — v_e, – – v_G, - - - - v_{LG}. (b) Logarithmic of MCR, for $\varepsilon = 1$. Denote: — v_e, – – v_G, - - - - v_{LG}.

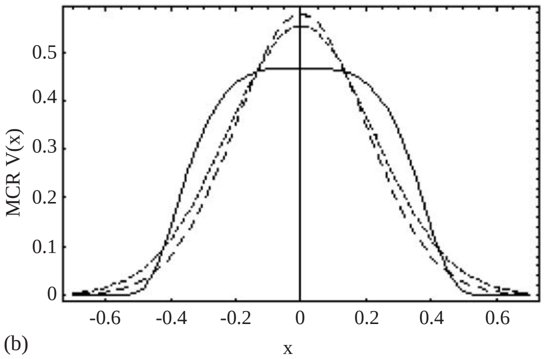
6. Conclusions

The most significant advantage of LOMSEC technique is to obtain much more improved solutions compared with using Caughey criterion, especially as strong non-linearity.

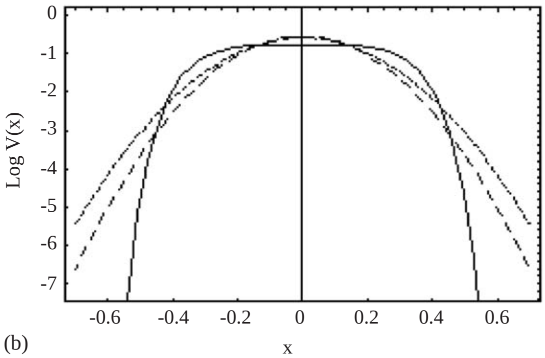
A defined value y_e^0 (integration domain) exists in LOMSEC for considered systems that leads to the exact solution. It means that in principle, it is possible for LOMSEC criterion to find exact solution, meanwhile this is impossible for Caughey criterion.

By the way of changing the limitation of integration domain, the LOMSEC provides with a series of different approximate solutions, and as $y^0 = \infty$ LOMSEC gives Caughey solution.

The averaging values y_a^0 for LOMSEC corresponding to the various systems are drawn from the investigation, which can be efficiently applied for the analysis of analogous non-linear random



(b)



(b)

Fig. 13. (a) MCR, for $\varepsilon = 100$. Denote: — v_e, – – v_G, - - - - v_{LG}. (b) Logarithmic of MCR, for $\varepsilon = 100$. Denote: — v_e, – – v_G, - - - - v_{LG}.

systems, especially for the analysis of moments of responses. Beside the systems presented in the paper, a series of other diverse non-linear random systems was also analyzed by the authors, which resulted in the averaging values $y_a^0 \in (2.1–2.7)$ corresponding to various systems, except $y_a^0 = 1.6$ for Vanderpol oscillator. For more convenient application of the proposed criterion, a fixed value y^0 applicable for any stochastic non-linear system should be recommended. By a series of examinations, the recommended value to be $y^0 = 2.5$.

Regarding the mean up-crossing rate, the results gained from the examples show that for a specific system, there exists a limited domain of the response where the AMCR by LOMSEC are better than that by Caughey criterion; and for larger values of the response than the limited domain, AMCR by Caughey and by LOMSEC are both generally not good at any non-linearity.

A main limitation of the proposed method is that no mathematical link between the expected integration and the system parameters is established. This means that for an arbitrary-specific system, we cannot determine the optimal integration domain, which allows to get the best approximate solution. In addition, by using the approximate integration domain as recommended, LOMSEC solution for some cases of weak non-linearity may be worse than that by Caughey.

Due to difficulty in the analytic calculation for multi-degree-of-freedom non-linear systems, LOMSEC only gives the ultimate result in the form of numerical which may be less accurate than some numerical methods, for example Runge–Kutta.

7. Summary

The criteria for constructing an equivalent linear system are usually based on the minimization of some specific deviation measure. This technique was first developed for deterministic non-linear problems. Caughey (1959) adapted this technique to apply to stochastic systems. The standard way of implementing this technique is to minimize a mean square measure of the difference between the non-linear and the equivalent linear equations. GEL proposed by Caughey is presently the simplest tool widely used for analysis of non-linear stochastic problems because GEL allows to use the available analytic results from stochastic linear systems. However, a major limitation of GEL is perhaps that its accuracy decreases as the non-linearity increases, and for many cases it can lead to unacceptable errors. Therefore, a series of researches for improving GEL has been done for the past some decades by many authors.

An alternative extension of GEL was proposed by Anh and Di Paola (1995). This extension is refereed to as 'local mean square error criterion' (LOMSEC). The Authors gave initial tests based on Duffing and Vanderpol oscillators under a zero mean Gaussian white noise. Following the initial efforts of Anh and Di Paola, Hung has recently examined the proposed technique through analysis of a series of diversely various non-linear random systems such as the analysis of the response moments of SDOF systems, the analysis of the mean up-crossing rate and the exceedance probability of response.

However, the theory of the proposed technique and the analyses given by the authors has been just demonstrated for non-linear random simple-degree-of-freedom (SDOF) systems. Therefore, this paper presents a comprehensive LOMSEC for non-linear random multi-degree-of-freedom (MDOF) systems. Thereupon, illustrative examples which include some SDOF and two-degree-of-freedom systems are given for demonstration. For comparison with Caughey's method through the evaluation of accuracy of the solutions, the systems selected for the analysis are ones for which there exist the known exact solution or the solution acknowledged as exact. The obtained results show efficient applicability of the improved criterion to approach more accurate solutions than that using the conventional linearization, especially for the analysis of mean square of the responses.

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Appendix

For a zero mean scalar Gaussian process y, all higher order moments $\langle y^{2n} \rangle$ can be expressed in terms of second order moments: $\langle y^{2n} \rangle = 1.3.5\ldots(2n-1)\langle y^2 \rangle^n$.

Analogously, all moments $[y^{2n}]$ in LOMSEC can also be expressed in terms of the second moments $\langle y^2 \rangle$ by the formula which is easily provable after replacing variable $y = t\sigma_y$:

$$[y^{2n}]_{-y^0\sigma_y}^{y^0\sigma_y} = 2T_{n,y^0}\langle y^2 \rangle^n, \quad n = 1, 2,$$

where

$$\sigma_y^{2n} = \langle y^2 \rangle^n, \quad T_{n,y^0} = \int_0^{y^0} t^{2n} n(t)\,\mathrm{d}t \quad \text{and} \quad n(t) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-t^2/2}.$$

Impute to n, y^0 concrete values, one gets T_{n,y^0} as a positive constant. In addition, all odd order moments are equal to zero.

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